Matematisk seminar
Universitetet i Oslo

No 4 - 1969

NOTE ON THE GEBA-GRANAS THEORY OF COMPACT FIELDS

by

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<u>Note on the Geba - Granas theory of compact fields.</u>

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This note is concerned with an extension of the Geba-Granas theory for compact fields in a Banach space [2]. It is related to the degree theory on Banach manifolds formulated by Smale and Elworthy-Tromba in much the same way as the Geba-Granas theory is related to the Leray-Schander degree , cf. [1], [3] and [4]. Given an arbitrary Banach space E and a paracompact Hausdorff space X we consider equivalence classes of proper "singularity free" maps $X \to E$, equivalent maps being compact perturbations of each other. If X is a manifold certain of these classes admits a degree function.

1. <u>Extension of the Geba - Granas theory.</u> Let E be a Banach space and $\{E_i\}$ a directed filtration of E by finite dimensional linear subspaces, such that any finite dimensional subspace of E is contained in some E_i . Let X be a paracompact Hausdorff space and $\varphi: X \to E$ a proper continuous map. Finally let $W \subset E$ be a closed subset contained in a finite dimensional subspace of E . (In the proofs we may assume without loss of generality that W is contained in all E_i .) A compact perturbation of $\varphi: X \to E$ which maps into $E - W$ will be called a

φ-map, and a compact perturbation of $\varphi \circ pr : X \times I \to E$ which maps into $E - W$ a φ-homotopy. If $h: X \times I \to E$ is a φ-homotopy we write $h = \varphi + H$ instead of $h = \varphi \circ pr + H$. At this stage we do not assume φ itself to be a φ-map, i.e. φ does not necessarily map into $E - W$.

Remark. A map $f: X \to E$ is a compact perturbation of φ if it can be written $\varphi + K$ for some compact map $K: X \to E$. A map $K: X \to E$ is compact if $\operatorname{im} K$ has compact closure in E.

Given a map $f: X \to E$ we can form the closed subsets $W_t = W_t(f) \subset W$ of elements $w \in W$ such that $\operatorname{dist}(w, \operatorname{im} f) \leq t$, t running through the non-negative real numbers. We say that f is W-bounded if W_t is bounded in E for all $t \geq 0$. This definition is introduced because it simultanously covers the two cases we are interested in: f bounded or W bounded. For reference we write down the following easy observation

Lemma 1.1 Let $\varphi: X \to E$ be a continuous map and $f: X \to E$ a compact perturbation of φ. If φ is proper, respectively W-bounded, so is f.

Proof. Let K be the compact map $f - \varphi$ and let $C = \overline{\operatorname{im} K}$. Take a compact subset $C' \subset E$. Then the vector space difference $C' - C$ is compact and so $\varphi^{-1}(C'-C)$ is compact, φ being proper. Since $f^{-1}C' \subset \varphi^{-1}(C'-C)$, $f^{-1}C'$ is also compact and so f is proper. If f was not W-bounded, we should have a sequence of elements $w_1, w_2, \ldots$ in W with $\|w_n\| > n$ and a sequence $x_1, x_2, \ldots$ in X such that $\operatorname{dist}(w_n, \varphi(x_n) + K(x_n)) \leq t$ for some $t \geq 0$. However, if c is the bound of K, this would emply $\operatorname{dist}(w_n, \varphi(x_n)) \leq t_o + c$, i.e. $w_n \in W_{t+c}(\varphi)$ contrary to the W-boundedness of φ .

Subsequently φ and W will be fixed having the properties already stated. φ is moreover supposed to be W-bounded. The next lemma is important, compare [ ] .

<u>Lemma 1.2</u> Let $h: X \times I \to E$ be a φ-homotopy. Then h is bounded away from , i.e. dist $(W \;\; \mathrm{im}\, h) > 0$.

<u>Proof</u>. If dist $(W, \mathrm{im}\, h) = 0$ there exists sequences $\{x_n\}$, $\{t_n\}$, $\{w_n\}$ such that $\|h(x_n, t_n) - w_n\| \leq 1/n$. But φ is W-bounded and therefore so is $\varphi \circ \mathrm{pr}$. By lemma 1 h is then W-bounded. Therefore $\{w_n\}$ sits in a bounded hence compact part of W and so may be assumed to converge to a point $w \in W$. Thus $\{h(x_n, t_n)\}$ converges to w . By lemma 1 h is proper, being a φ-homotopy, and so $\{(x_n, t_n)\}$ may be assumed to converge to a point $(x,t) \in X \times I$. By continuity we should have $h(x,t) = w$ contrary to the restriction $\mathrm{im}\, h \subset X - W$ on h .

<u>Corollary 1.3</u> Let $f: X \to E$ be a φ-map. Then f is bounded away from W .

<u>Corollary 1.4</u> Let $f: X \to E$ be a φ-map. Then f is φ-homotopic to map $f' = \varphi + K'$ with K' finite dimensional, and two such maps f' are φ-homotopic by a homotopy $h' = \varphi + H'$

<u>Proof</u>. Let $f = \varphi + K$, K compact. By corollary 1.3 there is $\epsilon > 0$ such that dist $(W, \mathrm{im}\, f) \dot{>} \epsilon$. Let $K' : X \to E$ be a compact finite dimensional map which ϵ-approximates K (cf. 4). Then $f' = \varphi + K'$ is a proper map which ϵ-approximates $f = \varphi + K$,

hence f' maps into $X - W$ and is a φ-map. The same is true for $(1-t)f + tf'$, $0 \leq t \leq 1$. Hence f is φ-homotopic to f'. Suppose f'_0, f'_1 are φ-maps of this type. There is an ordinary φ-homotopy $h = \varphi + H$ from f'_0 to f'_1. Again let $\epsilon > 0$ be such that $\operatorname{dist}(W, \operatorname{im} h) > \epsilon$, and let H'' be a compact finite dimensional ϵ-approximation to H. Then $h'' = \varphi + H''$ is an ϵ-approximation to h and so h'' maps into $X - W$ and is a φ-homotopy from f''_0 to f''_1, say. Finally let E_j be a finite dimensional subspace of E so big that $f'_0 - \varphi$, $f'_1 - \varphi$ and $h'' - \varphi$ all map into E_j. Then the composite of the homotopies $(1-t)f'_0 + tf''_0$, h'' and $(1-t)f''_1 + tf'_1$ is a φ-homotopy $h' = \varphi + H'$ from f'_0 to f'_1 with $H'(X \times I) \subset E_j$.

Next we introduce a suspension operation in the sets of finite dimensional φ-homotopy classes. For each index j write $X_j = \varphi^{-1}E_j$, and let $\varphi_j\colon X_j \to E_j$ be the map defined by φ. We assume $W \subset E_j$ for all j. Then φ_j is proper and W-bounded. First we note the following lemma, compare [2], theorems (2.1), (2.2):

<u>Lemma 1.5</u> Let $f\colon X_j \to E_j$ be a φ_j-map. Then there is an extension of f to a φ-map $f' = \varphi + K'$ with $\operatorname{im} K' \subset E_j$.

Let $f_0, f_1\colon X_j \to E_j$ be φ_j-homotopic φ_j-maps with φ-extensions $f'_0, f'_1\colon E \to X$ which differs from φ by compact maps into E_j. Let $h\colon X_j \times I \to E_j$ be a φ_j-homotopy from f_0 to f_1. Then there is an extension of h to a φ-homotopy $h' = \varphi + H'$ from f'_0 to f'_1 with $\operatorname{im} H' \subset E_j$.

<u>Proof</u>. Let $f = \varphi_j + K$, $K\colon X_j \to E_j$ compact (i.e. bounded). Let $K'\colon X \to E_j \subset E$ be a compact extension of K (Tietze's theorem) and define $f' = \varphi + K'$. Then f' actually maps into

$E - W$. In fact $f'(x) \in W \subset E_j$ implies $\varphi(x) \in E_j$ so that $x \in X_j$ and $f'(x) = f(x) \in E_j - W$, a contradiction. It follows that f' is a φ-extension of f . Again, let $h = \varphi_j + H$, $H: X_j \times I \to E_j$ compact, and let $f_0' = \varphi + K_0'$, $f_1' = \varphi + K_1'$ with $\operatorname{im} K_0' \subset E_j$, $\operatorname{im} K_1' \subset E_j$. Extend H to $(X \times 0) \cup (X_j \times I) \cup (X \times 1)$ by means of K_0', K_1' and further to some compact map $H': X \times I \to E_j \subset E$ by Tietze's theorem. Define $h' = \varphi + H'$. Then as above h' maps into $E - W$ and so is a φ-map .

Let $[X, E - W]_\varphi$ denote the set of φ-homotopy classes of φ-maps and $[X_j, E_j - W_j]_{\varphi_j}$ the corresponding set of φ_j-homotopy classes of φ_j-maps with singularity set $W_j = W \cap E_j$. (Then φ_j is W_j-bounded). Since ultimately $W_j = W$, we may apply lemma 5 and conclude that for "large" indices there is a canonical map

$$S_i : [X_i, E_i - W_i]_{\varphi_i} \to [X, E - W]_\varphi$$

If $j \geq i$ by choosing $X = X_j$, $E = E_j$ and $\varphi = \varphi_j$, we get a canonical map

$$S_{ji} : [X_i, E_i - W_i] \to [X_j, E_j - W_j]_{\varphi_j}$$

Also by lemma 5 follows that if $k \geq j \geq i$, then $S_{kj} S_{ji} = S_{ki}$ and $S_j S_{ji} = S_i$. Therefore the family $\{[X_i, E_i]_{\varphi_i}, S_{ji}\}$ form a direct system, and the family of maps $S_i : [X_i, E_i - W_i]_{\varphi_i} \to [X, E - W]_\varphi$ defines uniquely a map

$$S : \varinjlim [X_i, E_i - W_i]_{\varphi_i} \to [X, E - W]_\varphi$$

whose restriction to any $[X_i, E_i - W_i]_{\varphi_i}$ is S_i , i sufficiently large. The maps S_{ji} will be called suspensions. Observe now that by corollary 5 the limit map S is a bijection. Thus we have

Theorem 1.6 The set $[X,E-W]_\varphi$ of φ-homotopy classes of φ-maps is canonically isomorphic to the stable φ-homotopy set $\varinjlim [X_i, E_i - W_i]_{\varphi_i}$.

This generalizes the main result in [2], theorems 2,3, which treats the case where X is a closed bounded subset of E and φ is the inclusion map.

If φ is bounded, then the filtering spaces $X_i = \varphi^{-1}E_i$ are compact. In this case <u>any</u> continuous map $X_i \to E_i - W_i \subset E_i$ is a φ_i-map and any continuous homotopy $X_i \times I \to E_i - W_i \subset E_i$ is a φ_i-homotopy:

Corollary 1.7 If φ is bounded, the set $[X,E-W]_\varphi$ of φ-homotopy classes of φ-maps is canonically isomorphic to the stable homotopy set $\varinjlim [X_i, E_i - W_i]$.

Corollary 1.8 Let φ be bounded and let $W = \{o\}$. Then the set $[X, E - o]_\varphi$ is canonically isomorphic to the stable cohomotopy group $\varinjlim \pi^{d_i}(X_i)$, $d_i = \dim E_i - 1$.

Remark. These notions are easily relativized. Thus, let A be a closed subset of X and $\psi\colon X \to E$ a proper map which is W-bounded on A . Consider the ψ-maps $(X,A) \to (E, E-W)$, compact perturbations of ψ . (Thus A and $\psi|A\colon \to E$ plays the role of X and φ above.) There is a restriction map

$$[X,A;E,E-W]_\psi \to [A,E-W]_{\psi|A}$$

which is bijective since E is an absolute retract for paracompact spaces. Therefore we get directly relativized versions of theorem 6 and corollaries 7 and 8.

2. Applications to degree theory. We recall some of the basic definitions and properties connected with the degree of a Fredholm map. The reader is refered to [1] for a more complete discussion. Let $L(E)$ be the Banach algebra of bounded linear operators on E and $GL(E)$ the multiplicative subgroup of invertible elements. Let $c(E)$ be the completely continuous operators and $L_c(E)$ and $GL_c(E)$ the subsets of $L(E)$ and $GL(E)$, respectively, of operators of the form $I + T$, $T \in c(E)$. Then $GL_c(E)$ is a subgroup of $GL(E)$. It is known that $GL_c(E)$ has two components, cf. [1]. We denote the component containing the identity $SL_c(E)$ and the other $SL_c^-(E)$. Given a Banach manifold M a c-structure on M is an admissible atlas $\{\varphi_i, U_i\}$ maximal with respect to the property: For any i,j the differtial $d(\varphi_j\varphi_i^{-1})$ at any point lies in $GL_c(E)$. The c-structure is orientable if it admits a subatlas for which the differentials actually lie in $SL_c(E)$. An orientation is a subatlas maximal with respect to this property. Observe that any finite dimensional manifold has a unique c-structure and that orientability in this case has its usual meaning. A smooth map $f: M \to N$ between c-manifolds (i.e. manifolds with given c-structures) modelled on E is a c-map if for any local representative $\psi_j f \varphi_i^{-1}$ of f the differential $d(\psi_j f \varphi_i)$ at any point is in $L_c(E)$. This implies that f is Fredholm of index 0, i.e. the differential df is anywhere a linear Fredholm transformation with kernel and cokernel of the same dimension. Suppose f is a proper c-map between oriented manifolds M,N with N connected. Then the oriented degree of f is defined: By the Smale-Sard theorem f has a regular value y in N. Then $f^{-1}\{y\} \subset M$ consists of a finite number of points. Count these with their proper signs;

this gives the degree,

$$\deg f = \sum_{x\in f^{-1}\{y\}} \operatorname{sgn} df_x .$$

The sign (of f) at $x \in f^{-1}\{y\}$ is determined as follows: Take any local representative $\psi_j f \varphi_i^{-1}$ around x . The derivative $d(\psi_j f \varphi_i^{-1})$ at $\varphi_i(x)$ is then in $GL_c(E)$ since x is a regular point. Define $\operatorname{sgn} df_x$ to be 1 if $d(\psi_j f \varphi_i^{-1})$ is in $SL_c(E)$ and -1 otherwise. (The value does not depend on the choice of local representative.) This definition of degree obviously extends the finite dimensional one, cf. [4].

Suppose now that $N = U$, an open subset of E , with its canonical c-structure and that $f: M \to U$ is just Fredholm of index 0 . Then, by a result of Elworthy and Tromba [1], there is a unique admissible c-structure $c_f = \{\varphi_i, U_i\}$ on M making f a c-map. We will say that f is <u>orientable</u> if c_f is orientable. Then, if f is proper, the degree of f is defined, and it can be shown that up to sign it is a proper Fredholm homotopy invariant.

Returning to the end of section 1 we now assume that (X,A) is a smooth relative manifold modelled on E (i.e. A is a closed subset of X and $X-A$ is a smooth manifold modelled on E) and that W consists of the origin o only. We also assume that ψ maps A into $E-o$. Then ψ is itself a ψ-map ; in fact the collection of ψ-maps is an equivalence class with respect to the relation "compact perturbation of" in the set of all proper maps $X \to E$ sending A into $E-o$. The relation of ψ-homotopy further stratifies this equivalence class; more precicely, the relation of "compactly homotopic to" is a still finer equivalence relation in the set of proper maps. We aim to define a "compact homotopy" invariant degree on some of the perturbation classes, namely those which admit a smooth Fredholm representative of index 0 . Thus, assume henceforth that ψ is smooth and Fredholm of index 0 on $X - A$.

Lemma 2.1 Let (X,A) be a relative manifold modelled on a Banach space E and $\psi: (X,A) \to (E,E-o)$ a σ-proper map which is Fredholm on $X - A$. Let $\{E_i\}$ be a finite collection of finite dimensional subspaces of E . Then there are points $p \in E$ arbitrarily near the origin such that $\psi + p$ is transverse regular to all E_i on $\psi^{-1}W$, W any small neighbouhood of the origin in E .

Proof. For each i let E_i' be a complement of E_i in E such that we have split short exacts sequences of Banach spaces

$$0 \to E_i \to E \xrightarrow{pr_i} E_i' \to 0$$

and canonical isomorphisms $E_i \times E_i' \cong E$. Let W_i and W_i' be open neighbourhoods of the origin in E_i and E_i' , respectively, such that $W_i \times W_i' \subset E - \psi A$. Then for any i the composite

$$\psi^{-1}W_i \times W_i' \xrightarrow{\psi} W_i \times W_i' \xrightarrow{pr_i} W_i'$$

is a σ-proper Fredholm map, being compoed of such maps, and so its regular value set V_i' is residual in W_i' cf. [1]. It follows that the regular value set of the composite

$$\psi^{-1}W_i \times W_i' \xrightarrow{\psi} E \xrightarrow{pr_i} E_i'$$

being $V_i' \cup (E_i' - W_i')$ is residual in E_i' . It follows that the sets $pr_i^{-1}(V_i' \cup (E_i' - W_i'))$ are residual in E , hence their intersection V is dence. If $q \in V$, then $pr_i(q)$ is a regular value of $pr_i \circ \psi \mid \psi^{-1}W_i \times W_i'$ and so the origin is a regular value of $pr_i \circ (\psi - q) \mid \psi^{-1}W_i \times W_i'$ for any i. Thus the translate $\psi - q$ is transverse regular to all E_i on $\cap\psi^{-1}W_i \times W_i' = \psi^{-1}\cap W_i \times W_i'$

<u>Remark</u>. Since ψ is proper, ψA is closed and therefore bounded away from o. Hence for p sufficiently close to the origin, $\psi_p = \psi + p$ also maps A away from o and so does $\psi_{tp} = \psi + tp$ for $0 \leq t \leq 1$. It follows that for small p ψ is homotopic to $\psi + p : (X,A) \to (E,E-o)$ by a smooth compact one-dimensional homotopy $(X,A) \times I \to (E,E-o)$.

Assume from now on that the map $\psi : (X,A) \to (E,E-o)$ is proper and oriented (i.e. $\psi: X \to E$ is proper and $\psi|X - A$ oriented). Then the degree of ψ with respect to the origin is well defined : Let U be the (open) connected component of $E - \psi A$ containing the origin o, and let $V = \psi^{-1}U$. Then V is an open subset of X contained in $X - A$, and $\psi: V \to U$ is proper oriented Fredholm map of index 0. Hence its degree is defined. By definition this is the degree of ψ with respect to o, denoted $\deg(\psi,A,o)$. It is invariant under smooth compact homotopies $(X,A) \times I \to (E,E-o)$. Observe that if $B \subset E$ is a small open ball around the origin (an open neighbourhood in $E - \psi A$ would suffice) and $A' = X - \psi^{-1}B$, then $\deg(\psi,A,o) = \deg(\psi,A',o)$. Thus for degree purposes we may suppose that $X - A = \psi^{-1}B$, which makes $\psi: X - A \to B$ proper and $A = \psi^{-1}\psi A$. In the sequal we make repeatedly use of this.

One can show that if ψ is transverse regular (on $X - A$) to a finite dimensional subspace $E^n < E$, then $(\psi^{-1}E^n, \psi^{-1}E^n \cap A)$ is orientable, and that a specific orientation of c_f induces a specific orientation of $(\psi^{-1}E^n, \psi^{-1}E^n \cap A)$, cf. for instance [5]. Write (X^n, A^n) for $(\psi^{-1}E^n, \psi^{-1}E^n \cap A)$ and $\psi^n\colon (X^n, A^n) \to (E^n, E^n - o)$ for the map induced by ψ . Let $y \in E^n$ be a regular value for ψ^n close to the origin. Then y is a regular value for ψ and $\psi^{-1}\{y\} = (\psi^n)^{-1}\{y\}$. Since (X^n, A^n) inherits its orientation from (X,A) , $\operatorname{sgn} d\psi^n_x = \operatorname{sgn} d\psi_x$ for all $x \in \psi^{-1}\{y\}$. Thus $\deg(\psi, A, o) = \deg(\psi^n, A^n, o)$. However, $\deg(\psi^n, A^n, o)$ can be computed by well known homological methods: If $B \subset E$ is an open ball with $\psi^{-1}B = X - A$, let $\gamma \in H^n_c(E^n, E^n - B)$ be a generator (Čech cohomology with compact supports, coefficients $\mathbb{Z}$). Then $\deg(\psi^n, A^n, o)$ is up to sign the value on γ^n of the composite homomorphism

$$H^n_c(E^n, E^n-B) \xrightarrow{\psi^{n*}} H^n_c(X^n, A^n) \cong H_o(X^n - A^n) \xrightarrow{\varepsilon} \mathbb{Z}$$

In particular we can choose γ^n such that the homological degree comes out with the right sign. If $E^m \subset E^n$ are finite dimensional subspaces of E to which ψ is transverse regular, we get a diagram

$$\begin{array}{ccccccc}
H^n_c(E^n, E^n-B) & \xrightarrow{\psi^{n*}} & H^n_c(X^n, A^n) & \cong & H_o(X^n - A^n) & \xrightarrow{\varepsilon} & \mathbb{Z} \\
\cong \uparrow & & \uparrow & & \uparrow & & \| \\
H^m_c(E^m, E^m-B) & \xrightarrow{\psi^{m*}} & H^m_c(X^m, A^m) & \cong & H_o(X^m - A^m) & \xrightarrow{\varepsilon} & \mathbb{Z}
\end{array}$$

where $H^m_c(E^m, E^m-B) \to H^n_c(E^n, E^n-B)$ is the suspension or the Thom isomorphism of the normal bundle of E^m in E^n sending γ^m to γ^n , and $H^m_c(X^m, A^m) \to H^n_c(X^n, A^n)$ is the induced Thom map. The latter as well as the duality isomorphisms are best interpreted by using the excisions $H^*_c(X^i, A^i) \cong H^*_c(X^i - A^i)$.

Let $g\colon (X,A) \to (E, E-o)$ be a compact finite dimensional (but not necessarily smooth) perturbation of ψ , say $g = \psi + K'$.

Let $E^m \subset E$ be any finite dimensional subspace of E containing the image of K'. Translate ψ slightly to make it transverse regular to E^m. This gives us a translate $g_p = \psi_p + K'$ and a homotopy $g_{tp} = \psi_{tp} + K'$ (cf. remark to lemma 2.1). Since $\operatorname{im} K' \subset E^n$, we have $(g_p^{-1}E^m, A \cap g_p^{-1}E^m) = (\psi_p^{-1}E^m, A \cap \psi_p^{-1}E^m)$. With an extention of our earlier notation we write this pair (X_p^m, A_p^m) so that we have the maps

$$H_c^m(E^m, E^m - B) \xrightarrow{g_p^{m*}} H_c^m(X_p^m, A_p^m) \overset{D}{\cong} H_o(X_p^m - A_p^m) \xrightarrow{\varepsilon} \mathbb{Z}$$

.

<u>Lemma 2.2</u> If E^m, E^n both contain $\operatorname{im} K'$ and g_p, g_q are small translations transverse regular to E^m, E^n respectively, then

$$\varepsilon D g_p^{m*}(\gamma^m) = \varepsilon D g_q^{n*}(\gamma^n)$$

<u>Proof</u>. First assume $E^n = E^m$. There is a compact finite dimensional homotopy $G : (X,A) \times I \to (E, E-o)$ from ψ_p to ψ_q which is a Fredholm map of index 1. Let $I \to I$ be a smooth map, strictly positive on $(0,1)$, and such that it and all its derivatives tend to 0 at the boundary $\{0,1\}$. Set $\varepsilon : X \times I \xrightarrow{pr} I \to I$. Then $G' = \frac{1}{\varepsilon} G$ is Fredholm of index 1 on $X \times (0,1)$. By Lemma 2.1 there is a point $r \in E$ close to the origin such that $\frac{1}{\varepsilon}G + r$ is transverse regular to E^m on $(X - A) \times (0,1)$. It follows that $G + \varepsilon \cdot r$ is transverse regular to E^m on $(X - A) \times I$ and homotopic to G by the smooth compact finite dimensional homotopy $G + \tau \cdot \varepsilon \cdot r$, $0 \leq \tau \leq 1$. Pulling back E^m by $G_{\varepsilon \cdot r} = G + \varepsilon \cdot r$ gives a relative manifold (Y^{m+1}, C^{m+1}) and a commutative diagram

$$
\begin{tikzcd}
H_c^m(E^m, E^m-B) \arrow[r, "G_{\varepsilon r}^{m*}"] \arrow[dr, "(g_p^m+g_q^m)*"] \arrow[ddr, "g_p^{m*}"'] & H_c^m(Y^{m+1}, C^{m+1}) \arrow[r, phantom, "\overset{D}{\cong}"] \arrow[d] & H_1(Y^{m+1}-C^{m+1}, (X_p^m-A_p^m) \cup (X_q^m-A_q^m)) \arrow[d, dashed] \\
 & H_c^m((X_p^m, A_p^m) + (X_q^m, A_q^m)) \arrow[r, phantom, "\overset{D}{\cong}"] \arrow[d] & H_o((X_p^m, A_p^m) + (X_q^m, A_q^m)) \arrow[d, dashed] \\
 & H_c^m(X_p^m, A_p^m) \arrow[r, phantom, "\cong"] & H_o(X_p^m, A_p^m)
\end{tikzcd}
$$

The claim now follows from the diagram.

Suppose next that $E^n \neq E^m$. In that case we may as well suppose $E^m \subset E^n$. If g_q is not transverse regular also to E_m , replase q with an element also denoted q for which this is true. According to the first part of the proof this does not change the value of $\varepsilon D g_q^{n*}(Y^n)$. The conclusion now follows.

This completes the proof of lemma 2.2.

We can now define the degree of g with respect to the origin by

$$\deg\,(g, A, \ ) \;=\; \deg\,(g_p^m, A_p^m, o)$$

By lemma 2.2 the degree is well defined. It is easy to see that this does not depend on the particular choise of smooth representative ψ in the perturbation class. The degree function so defined is an extension of the ordinary one for cmooth Fredholm maps of index 0 and of the Leray-Schauder degree (of not necessatily smooth maps), and satisfies conditions analogous to the Leray-Schauder degree, cf. [4] p. 86.

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